On the idempotents of Hecke algebras

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Abstract

We give a new construction of primitive idempotents of the Hecke algebras associated with the symmetric groups. The idempotents are found as evaluated products of certain rational functions thus providing a new version of the fusion procedure for the Hecke algebras. We show that the normalization factors which occur in the procedure are related to the Ocneanu–Markov trace of the idempotents.

1 Introduction

It was observed by Jucys [8] that the primitive idempotents of the symmetric group $\mathfrak{S}_n$ can be obtained by taking certain limit values of the rational function

$$\Phi(u_1, \dots, u_n) = \prod_{1 \leqslant i < j \leqslant n} \Big(1 - \frac{(i\,j)}{u_i - u_j}\Big), \tag{1}$$

where $u_1, \dots, u_n$ are complex variables and the product is calculated in the group algebra $\mathbb{C}[\mathfrak{S}_n]$ in the lexicographical order on the pairs (i, j). A similar construction, now commonly referred to as the *fusion procedure*, was developed by Cherednik [1], while complete proofs were given by Nazarov [13]. A simple version of the fusion procedure establishing its equivalence with the Jucys–Murphy construction was recently found by one of us in [10]; see also [11, Ch. 6] for applications to the Yangian representation theory and more references. In more detail, let $\mathcal{T}$ be a standard tableau associated with a partition λ of n and let $c_k = j - i$, if the element k occupies the cell of the tableau in row i and column j. Then the consecutive evaluations

$$\Phi(u_1, \dots, u_n)\big|_{u_1=c_1}\big|_{u_2=c_2} \dots \big|_{u_n=c_n}$$

are well-defined and this value yields the corresponding primitive idempotent $E^{\lambda}_{\mathcal{T}}$ multiplied by the product of the hooks of the diagram of λ. The left ideal $\mathbb{C}[\mathfrak{S}_n]\, E^{\lambda}_{\mathcal{T}}$ is the

irreducible representation of $\mathfrak{S}_n$ associated with λ, and the $\mathfrak{S}_n$-module $\mathbb{C}[\mathfrak{S}_n]$ is the direct sum of the left ideals over all partitions λ and all λ-tableaux $\mathcal{T}$.

Our aim in this paper is to derive an analogous version of the fusion procedure for the Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(q)$ associated with $\mathfrak{S}_n$. The procedure goes back to Cherednik [2], while detailed proofs relying on q-versions of the Young symmetrizers were given by Nazarov [14]; see also Grime [4] for its hook version. We use a different approach based on the formulas for the primitive idempotents of $\mathcal{H}_n$ in terms of the *Jucys–Murphy elements*. These formulas derived by Dipper and James [3] generalize the results of Jucys [9] and Murphy [12] for $\mathfrak{S}_n$.

The main result of this paper is an explicit formula for the orthogonal primitive idempotents of $\mathcal{H}_n$. The idempotents are obtained as a result of consecutive evaluations of a rational function similar to (1). The normalization factors in the expressions for the Hecke algebra idempotents turn out to be related to the Ocneanu–Markov trace of the idempotents.

2 Idempotents of $\mathcal{H}_n$ and Jucys–Murphy elements

Let q be a formal variable. The Hecke algebra $\mathcal{H}_n$ over the field $\mathbb{C}(q)$ is generated by the elements $T_1, \dots, T_{n-1}$ subject to the defining relations

$$\begin{aligned} T_i\, T_{i+1}\, T_i &= T_{i+1}\, T_i\, T_{i+1}, \\ T_i\, T_j &= T_j\, T_i \quad \text{for} \quad |i-j|>1, \\ T_i^2 &= 1 + (q-q^{-1})\, T_i. \end{aligned}$$

Given a reduced decomposition $w = s_{i_1} \dots s_{i_l}$ of an element $w \in \mathfrak{S}_n$ in terms of the generators $s_i = (i, i+1)$, set $T_w = T_{i_1} \dots T_{i_l}$. Then T_w does not depend on the reduced decomposition, and the set $\{T_w \mid w \in \mathfrak{S}_n\}$ is a basis of $\mathcal{H}_n$ over $\mathbb{C}(q)$.

The *Jucys-Murphy elements* $y_1, \dots, y_n$ of $\mathcal{H}_n$ are defined inductively by

$$y_1 = 1, \qquad y_{k+1} = T_k\, y_k\, T_k \qquad \text{for} \quad k = 1, \dots, n-1. \tag{2}$$

These elements satisfy

$$y_k\, T_m = T_m\, y_k, \qquad m \neq k, k-1.$$

In particular, $y_1, \dots, y_n$ generate a commutative subalgebra of $\mathcal{H}_n$. The elements y_k can be written in terms of the elements $T_{(i\,j)} \in \mathcal{H}_n$, associated with the transpositions $(i\,j) \in \mathfrak{S}_n$ as follows:

$$y_k = 1 + (q-q^{-1})\,\big(T_{(1\,k)} + T_{(2\,k)} + \dots + T_{(k-1\,k)}\big).$$

Hence, the normalized elements $(y_k-1)/(q-q^{-1})$ specialize to the Jucys–Murphy elements for $\mathfrak{S}_n$ as $q \to 1$; see [9], [12], [3].

For any $k = 1, \dots, n$ we let w_k denote the unique longest element of the symmetric group $\mathfrak{S}_k$ which is regarded as the natural subgroup of $\mathfrak{S}_n$. The corresponding elements $T_{w_k} \in \mathcal{H}_n$ are then given by $T_{w_1} = 1$ and

$$T_{w_k} = T_1(T_2\, T_1) \cdots (T_{k-2} \dots T_1)(T_{k-1} T_{k-2} \dots T_1) \tag{3}$$

$$= (T_1 \dots T_{k-2}\, T_{k-1})(T_1 \dots T_{k-2}) \cdots (T_1\, T_2)\, T_1, \qquad k = 2, \dots, n. \tag{4}$$

We point out the following properties of the elements T_{w_k} which are easily verified by induction with the use of (3) and (4):

$$
\begin{aligned}
T_{w_k}\, T_j &= T_{k-j}\, T_{w_k}, \qquad & 1 \leqslant j < k \leqslant n, \\
T_{w_k}^2 &= y_1 y_2 \cdots y_k, \qquad & k = 1, \dots, n.
\end{aligned}
\tag{5}
$$

Following [14], for each $i = 1, \dots, n-1$ set

$$
T_i(x, y) = \frac{T_i\, y - T_i^{-1}\, x}{y - x} = T_i + \frac{q - q^{-1}}{x^{-1} y - 1},
\tag{6}
$$

where x and y are complex variables. We will regard the $T_i(x, y)$ as rational functions in x and y with values in $\mathcal{H}_n$. It is well-known that they satisfy the relations

$$
T_i(x, y)\, T_{i+1}(x, z)\, T_i(y, z) = T_{i+1}(y, z)\, T_i(x, z)\, T_{i+1}(x, y),
\tag{7}
$$

(the Yang–Baxter equation), and

$$
T_i(x, y)\, T_i(y, x) = \frac{(x - q^2 y)\,(x - q^{-2} y)}{(x - y)^2}.
\tag{8}
$$

Lemma 2.1. *We have the identities*

$$
T_{w_k}\, T_j(x, y) = T_{k-j}(x, y)\, T_{w_k}, \qquad 1 \leqslant j < k \leqslant n,
\tag{9}
$$

and

$$
T_{w_{k+1}}\, T_2(u, \sigma_{k-1}) \dots T_k(u, \sigma_1)\, T_{w_k}^{-1} = T_{w_k}\, T_1(u, \sigma_{k-1}) \dots T_{k-1}(u, \sigma_1)\, T_{w_{k-1}}^{-1} T_k,
\tag{10}
$$

where $1 \leqslant k < n$ *and* $u, \sigma_1, \dots, \sigma_{k-1}$ *are complex parameters.*

Proof. Relation (9) is immediate from (5), while (10) is deduced from

$$
(T_k \dots T_2 T_1) T_j(x, y) = T_{j-1}(x, y)\, (T_k \dots T_2 T_1), \qquad 2 \leqslant j \leqslant k,
$$

by taking into account the identity

$$
T_{w_k}^{-1} T_{w_{k+1}} = T_{w_{k-1}}^{-1} T_k T_{w_k} = T_k \dots T_2 T_1
$$

implied by (3) and (4). ∎

Now we recall the construction of the orthogonal primitive idempotents for the Hecke algebra from [3]. We will identify a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ of n with its diagram which is a left-justified array of rows of cells such that the top row contains λ_1 cells, the next row contains λ_2 cells, etc. A cell outside λ is called *addable* to λ if the union of λ and the cell is a diagram. A tableau $\mathcal{T}$ of shape λ (or a λ-tableau $\mathcal{T}$) is obtained by filling in the cells of the diagram bijectively with the numbers $1, \dots, n$. A tableau $\mathcal{T}$ is called standard if its entries increase along the rows and down the columns. If a cell occurs in row i and column j, its *q-content* will be defined as $q^{2(j-i)}$.

In accordance to [3], a set of orthogonal primitive idempotents $\{E^{\lambda}_{\mathcal{T}}\}$ of $\mathcal{H}_n$, parameterized by partitions λ of n and standard λ-tableaux $\mathcal{T}$ can be constructed inductively by the following rule. Set $E^{\lambda}_{\mathcal{T}} = 1$ if $n = 1$, whereas for $n \geqslant 2$,

$$E^{\lambda}_{\mathcal{T}} = E^{\mu}_{\mathcal{U}} \frac{(y_n - \rho_1) \dots (y_n - \rho_k)}{(\sigma - \rho_1) \dots (\sigma - \rho_k)}, \tag{11}$$

where $\mathcal{U}$ is the tableau obtained from $\mathcal{T}$ by removing the cell α occupied by n, μ is the shape of $\mathcal{U}$, and $\rho_1, \dots, \rho_k$ are the q-contents of all addable cells of μ except for α, while σ is the q-content of the latter. In particular, if λ and λ' are partitions of n, then

$$E^{\lambda}_{\mathcal{T}} \, E^{\lambda'}_{\mathcal{T}'} = \delta_{\lambda\lambda'} \delta_{\mathcal{T}\mathcal{T}'} \, E^{\lambda}_{\mathcal{T}}$$

for arbitrary standard tableaux $\mathcal{T}$ and $\mathcal{T}'$ of shapes λ and λ', respectively. Moreover,

$$\sum_{\lambda} \sum_{\mathcal{T}} E^{\lambda}_{\mathcal{T}} = 1,$$

summed over all partitions λ of n and all standard λ-tableaux $\mathcal{T}$.

In what follows we will omit the superscript λ and write simply $E_{\mathcal{T}}$ instead of $E^{\lambda}_{\mathcal{T}}$. Given a standard λ-tableau $\mathcal{T}$ and $k \in \{1, \dots, n\}$, we set $\sigma_k = q^{2(j-i)}$ if the element k of $\mathcal{T}$ occupies the cell in row i and column j. Then

$$y_k \, E_{\mathcal{T}} = E_{\mathcal{T}} \, y_k = \sigma_k \, E_{\mathcal{T}}. \tag{12}$$

Furthermore, given a standard tableau $\mathcal{U}$ with $n - 1$ cells, the corresponding idempotent $E_{\mathcal{U}}$ can be written as

$$E_{\mathcal{U}} = \sum_{\mathcal{T}} E_{\mathcal{T}}, \tag{13}$$

summed over all standard tableaux $\mathcal{T}$ obtained from $\mathcal{U}$ by adding one cell with entry n. Exactly as in the case of the symmetric group $\mathfrak{S}_n$ (see [10]), this relation can be used to derive the following alternative form of (11). Consider the rational function

$$E_{\mathcal{T}}(u) = E_{\mathcal{U}} \, \frac{u - \sigma_n}{u - y_n} \tag{14}$$

in a complex variable u with values in $\mathcal{H}_n$. Then this function is regular at $u = \sigma_n$ and the corresponding value coincides with $E_{\mathcal{T}}$:

$$E_{\mathcal{T}} = E_{\mathcal{U}} \, \frac{u - \sigma_n}{u - y_n}\Big|_{u=\sigma_n}. \tag{15}$$

3 Fusion formulas for primitive idempotents

For $k = 1, \dots, n - 1$ introduce the elements of $\mathcal{H}_n$ by

$$Y_k(\sigma_1, \sigma_2, \dots, \sigma_k; u) = T_{w_k} \, T_k(\sigma_1, u) T_{k-1}(\sigma_2, u) \dots T_1(\sigma_k, u) \, T^{-1}_{w_{k+1}}, \tag{16}$$

where $\sigma_1, \sigma_2, \dots, \sigma_k$ and u are complex parameters.

Lemma 3.1. *Let $\mathcal{U}$ be a standard tableau with k cells and the q-contents $\sigma_1, \sigma_2, \dots, \sigma_k$. Then*

$$E_{\mathcal{U}}\, Y_k(\sigma_1, \dots, \sigma_k; u) = \\ = (u - \sigma_1) \left(\prod_{j=1}^{k} \frac{(u - q^2\sigma_j)\,(u - q^{-2}\sigma_j)}{(u - \sigma_j)^2} \right) E_{\mathcal{U}} (u - y_{k+1})^{-1}. \tag{17}$$

Proof. We start with representing (17) in the form

$$(u - \sigma_1)^{-1}\, E_{\mathcal{U}}(u - y_{k+1}) = E_{\mathcal{U}}\, T_{w_{k+1}}\, T_1(u, \sigma_k) \dots T_k(u, \sigma_1)\, T_{w_k}^{-1}, \tag{18}$$

where we have used (8) and taken into account the fact that $E_{\mathcal{U}}$ commutes with y_{k+1}. Now we prove (18) by induction. For $k = 1$ we have

$$(u - \sigma_1)^{-1}(u - T_1^2) = T_1 \cdot T_1(u, \sigma_1),$$

which is true, as $\sigma_1 = 1$. Due to (9) and (10), the right hand side of (18) can be written in the form

$$E_{\mathcal{U}}\, T_k(u, \sigma_k)\, T_{w_{k+1}}\, T_2(u, \sigma_{k-1}) \dots T_k(u, \sigma_1)\, T_{w_k}^{-1} = \\ = E_{\mathcal{U}}\, T_k(u, \sigma_k)\, T_{w_k}\, T_1(u, \sigma_{k-1}) \dots T_{k-1}(u, \sigma_1)\, T_{w_{k-1}}^{-1} T_k.$$

Using (13), we can write $E_{\mathcal{U}} = E_{\mathcal{U}} E_{\mathcal{V}}$, where $\mathcal{V}$ is the tableau obtained from $\mathcal{U}$ by removing the cell occupied by k. Hence, the right hand side of (18) becomes

$$E_{\mathcal{U}} E_{\mathcal{V}}\, T_k(u, \sigma_k)\, T_{w_k}\, T_1(u, \sigma_{k-1}) \dots T_{k-1}(u, \sigma_1)\, T_{w_{k-1}}^{-1} T_k = \\ = E_{\mathcal{U}}\, T_k(u, \sigma_k) \left(E_{\mathcal{V}} T_{w_k}\, T_1(u, \sigma_{k-1}) \dots T_{k-1}(u, \sigma_1)\, T_{w_{k-1}}^{-1} \right) T_k = \\ = (u - \sigma_1)^{-1}\, E_{\mathcal{U}} T_k(u, \sigma_k)(u - y_k) T_k.$$

The last equality holds by the induction hypothesis. Now we represent $T_k(u, \sigma_k)$ in the form

$$T_k(u, \sigma_k) = \frac{T_k\, \sigma_k - T_k^{-1}\, u}{\sigma_k - u} = T_k + \frac{(q - q^{-1})\, u}{\sigma_k - u}.$$

This gives

$$E_{\mathcal{U}} T_k(u, \sigma_k)(u - y_k)\, T_k = E_{\mathcal{U}} \left(T_k + \frac{(q - q^{-1})\, u}{\sigma_k - u} \right) (u - y_k)\, T_k = \\ = E_{\mathcal{U}} \left(-u\, (q - q^{-1})\, T_k + u\, T_k^2 - y_{k+1} \right) = E_{\mathcal{U}} \left(u - y_{k+1} \right),$$

thus completing the proof. ■

Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of n. We will use the conjugate partition $\lambda' = (\lambda'_1, \dots, \lambda'_m)$ so that λ'_j is the number of cells in the j-th column of λ. If $\alpha = (i, j)$ is a cell of λ, then the corresponding hook is defined as $h_\alpha = \lambda_i + \lambda'_j - i - j + 1$ and the content is $c_\alpha = j - i$. Set

$$f(\lambda) = \prod_{\alpha \in \lambda} \frac{q^{\,c_\alpha}}{[h_\alpha]_q}, \tag{19}$$

where we have used the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

Suppose that $\mathcal{T}$ is a standard λ-tableau. As before, for each $k \in \{1,\dots,n\}$ we let σ_k denote the q-content $q^{2(j-i)}$ of the cell (i,j) occupied by k in $\mathcal{T}$. Consider the rational function

$$F_n(u) = \frac{u-\sigma_n}{u-\sigma_1}\prod_{k=1}^{n-1}\frac{(u-\sigma_k)^2}{(u-q^2\sigma_k)(u-q^{-2}\sigma_k)}.$$

Lemma 3.2. *The rational function $F_n(u)$ is regular at $u=\sigma_n$ and*

$$F_n(\sigma_n) = f(\mu)^{-1}\, f(\lambda),$$

where μ denotes the shape of the standard tableau obtained from $\mathcal{T}$ by removing the cell occupied by n.

Proof. It is clear that $F_n(u)$ depends only on the shape μ and does not depend on the standard tableau $\mathcal{U}$ obtained from $\mathcal{T}$ by removing the cell occupied by n. Therefore, we may assume that $\mathcal{U}$ is the row tableau obtained by writing the elements $1,\dots,n-1$ into the cells of μ consecutively by rows starting with the top row. Suppose that the rows of μ are

$$\mu_1 = \dots = \mu_{p_1} > \mu_{p_1+1} = \dots = \mu_{p_2} > \dots > \mu_{p_{s-1}+1} = \dots = \mu_{p_s}$$

for some integers $p_1,\dots,p_s$ such that $1 \leqslant p_1 < p_2 < \dots < p_s$ and some $s \geqslant 1$. With this notation, $F_n(u)$ can be written in the form

$$F_n(u) = (u-\sigma_n)\prod_{i=1}^{s}(u-q^{2\,\mu_{p_i}-2\,p_i})\prod_{i=0}^{s}(u-q^{2\,\mu_{p_i+1}-2\,p_i})^{-1},$$

where we set $p_0=0$ and $\mu_{p_s+1}=0$. Possible values of the q-content σ_n are $\sigma_n = q^{2\,\mu_{p_j+1}-2\,p_j}$ for $j=0,1,\dots,s$. Hence, for a fixed value of j the factor $u-\sigma_n$ cancels, and so $F_n(\sigma_n)$ is well-defined and can be expressed in the form

$$F_n(\sigma_n) = \big(q^{2\,\mu_{p_j+1}} - q^{2\,\mu_{p_j+1}+2}\big)\prod_{\alpha\in\mu}(1-q^{2h_\alpha})\prod_{\alpha\in\lambda}(1-q^{2h_\alpha})^{-1}, \tag{20}$$

which is verified by a simple calculation. On the other hand, $f(\lambda)$ can be represented as

$$f(\lambda) = q^{b(\lambda)}\,(1-q^2)^n\prod_{\alpha\in\lambda}(1-q^{2h_\alpha})^{-1}, \qquad b(\lambda) = \sum_{i\geqslant 1}\lambda_i(\lambda_i-1).$$

Therefore, the expression in (20) equals $f(\mu)^{-1}\, f(\lambda)$, as required. ∎

Introduce the rational function $\Psi(u_1,\dots,u_n)$ in complex variables $u_1,\dots,u_n$ with values in $\mathcal{H}_n$ by the formula

$$\Psi(u_1,\dots,u_n) = \overrightarrow{\prod_{k=1,\dots,n-1}}\Big(T_k(u_1,u_{k+1})\,T_{k-1}(u_2,u_{k+1})\dots T_1(u_k,u_{k+1})\Big)\cdot T_{w_n}^{-1}.$$

As before, we let λ be a partition of n and let $\mathcal{T}$ be a standard λ-tableau.

Theorem 3.3. *The idempotent $E_{\mathcal{T}}$ can be obtained by the consecutive evaluations*

$$E_{\mathcal{T}} = f(\lambda)\cdot \Psi(u_1,\dots,u_n)\big|_{u_1=\sigma_1}\big|_{u_2=\sigma_2}\cdots\big|_{u_n=\sigma_n}, \tag{21}$$

where the rational functions are regular at the evaluation points at each step.

Proof. We argue by induction on n. For $n \geqslant 2$ we let $\mathcal{U}$ denote the standard tableau obtained from $\mathcal{T}$ by removing the cell occupied by n and let μ be the shape of $\mathcal{U}$. Applying Lemma 3.2 and the induction hypothesis, we can write the right hand side of (21) in the form

$$F_n(\sigma_n)\, E_{\mathcal{U}}\, Y_{n-1}(\sigma_1,\dots,\sigma_{n-1};u_n)\big|_{u_n=\sigma_n},$$

where the elements $Y_{n-1}(\sigma_1,\dots,\sigma_{n-1};u_n)$ are defined in (16). The proof is completed by the application of Lemma 3.1 and relation (15). ■

Example 3.4. Using (21), for $n=3$ and $\lambda=(2,1)$ we get

$$E_{\mathcal{T}} = \frac{1}{[3]_q} T_1(\sigma_1,\sigma_2)T_2(\sigma_1,\sigma_3)T_1(\sigma_2,\sigma_3)(T_1T_2T_1)^{-1}. \tag{22}$$

In particular,

$$\sigma_1 = 1,\quad \sigma_2 = q^2,\quad \sigma_3 = q^{-2} \qquad \text{for} \qquad \mathcal{T} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and

$$\sigma_1 = 1,\quad \sigma_2 = q^{-2},\quad \sigma_3 = q^{2} \qquad \text{for} \qquad \mathcal{T} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,.$$

Note that (22) can be reduced to the fusion formulas contained in [5, p. 106]. ■

Example 3.5. For $n=4$ and $\lambda=(2^2)$ the idempotent $E_{\mathcal{T}}$ is obtained by evaluating the rational function

$$\frac{1}{[3]_q[2]_q^2}\, T_1(u_1,u_2)T_2(u_1,u_3)T_1(u_2,u_3)T_3(u_1,u_4)T_2(u_2,u_4)T_1(u_3,u_4)\, T_{w_4}^{-1} \tag{23}$$

consecutively at $u_1=\sigma_1$, $u_2=\sigma_2$, $u_3=\sigma_3$, and $u_4=\sigma_4$. We have

$$\sigma_1 = 1,\quad \sigma_2 = q^2,\quad \sigma_3 = q^{-2},\quad \sigma_4 = 1 \qquad \text{for} \qquad \mathcal{T} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

and

$$\sigma_1 = 1,\quad \sigma_2 = q^{-2},\quad \sigma_3 = q^{2},\quad \sigma_4 = 1 \qquad \text{for} \qquad \mathcal{T} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\,.$$

Note that for both tableaux the expression (23) contains the factor $T_3(u_1,u_4)$ which is not defined for $u_1=\sigma_1$ and $u_4=\sigma_4$. Nevertheless, the whole expression (23) is regular under the consecutive evaluations due to Theorem 3.3. We will use this example to illustrate the relationship with the approach of [14]. Using the relation (7) one can rewrite (23) as

$$\frac{1}{[3]_q[2]_q^2}\, T_2(u_2,u_3)T_1(u_1,u_3)T_2(u_1,u_2)T_3(u_1,u_4)T_2(u_2,u_4)T_1(u_3,u_4)\, T_{w_4}^{-1}.$$

By [14, Lemma 2.1], the product $T_2(u_1,u_2)T_3(u_1,u_4)T_2(u_2,u_4)$ is equal to

$$\frac{\big((T_2u_2 - T_2^{-1}u_1)\,T_3\,(T_2u_4 - T_2^{-1}u_2) + (q-q^{-1})\,u_1((q-q^{-1})\,u_2T_2 + u_2 - u_1)\big)}{(u_2-u_1)(u_4-u_2)} - \frac{(q-q^{-1})\,u_1(u_1-q^2u_2)(u_1-q^{-2}u_2)}{(u_2-u_1)(u_4-u_1)(u_4-u_2)}$$

and it is regular for $u_1 = q^{\pm 2}u_2$ at $u_1 = u_4$. It was shown in [14] that such considerations can be extended to the general expression (21) to prove that it is regular in the limits $u_i \to \sigma_i$. ∎

We conclude this section by showing that taking an appropriate limit in Theorem 3.3 as $q \to 1$ we can recover the respective formulas of [10] for the primitive idempotents of the symmetric group $\mathfrak{S}_n$.

Take the parameters x and y in (6) in the form $x = q^{2u}$ and $y = q^{2v}$. Since $T_i \xrightarrow[q\to 1]{} s_i$, for the limit value of $T_i(x,y)$ we have

$$T_i(x,y) = T_i + \frac{q^{u-v}}{[v-u]_q} \xrightarrow[q\to 1]{} s_i\,\varphi_{i,i+1}(u,v), \tag{24}$$

where

$$\varphi_{i,j}(u,v) = 1 - \frac{(i\,j)}{u-v}.$$

Using (24) we can calculate the corresponding limit for the element (16) to get

$$Y_k(\sigma_1,\sigma_2,\dots,\sigma_k;u) \xrightarrow[q\to 1]{} \varphi_{1,k+1}(c_1,u)\varphi_{2,k+1}(c_2,u)\dots\varphi_{k,k+1}(c_k,u), \tag{25}$$

where $\sigma_m = q^{2c_m}$. Clearly, the normalization factor $f(\lambda)$ specializes to the inverse of the product of the hooks of λ, and so the substitution of (25) into (21) leads to the main result of [10].

4 The Ocneanu–Markov trace of the idempotents

The purpose of this section is to calculate the Ocneanu–Markov trace of the idempotents $E_{\mathcal{T}}$ which turns out to be related to the normalization factor $f(\lambda)$ defined in (19).

Definition 4.1. For any given standard tableau $\mathcal{T}$ with n cells, its *quantum dimension* is defined as

$$\operatorname{qdim}\mathcal{T} = \mathcal{T}r^n(E_{\mathcal{T}})\,, \tag{26}$$

where $\mathcal{T}r^n : \mathcal{H}_n \to \mathbb{C}$ is the Ocneanu–Markov trace; see e.g. [7]. ∎

The Ocneanu–Markov trace $\mathcal{T}r^n$ can be defined as the composition of the maps

$$\mathcal{T}r^n = \mathrm{Tr_1 Tr_2 \dots Tr_n}.$$

The linear maps $\mathrm{Tr_{m+1}} : \mathcal{H}_{m+1} \to \mathcal{H}_m$ from the Hecke algebra $\mathcal{H}_{m+1}$ to its natural subalgebra $\mathcal{H}_m$ are determined by the following properties, where $Q \in \mathbb{C}$ is a fixed parameter, while $X, Y \in \mathcal{H}_m$ and $Z \in \mathcal{H}_{m+1}$:

$$\begin{aligned}&\mathrm{Tr_{m+1}}(XZY) = X\,\mathrm{Tr_{m+1}}(Z)Y, \quad \mathrm{Tr_{m+1}}(X) = QX,\\ &\mathrm{Tr_{m+1}}(T_m^{\pm 1} X T_m^{\mp 1}) = \mathrm{Tr_m}(X), \quad \mathrm{Tr_{m+1}}(T_m) = 1,\\ &\mathrm{Tr_m}\mathrm{Tr_{m+1}}(T_m Z) = \mathrm{Tr_m}\mathrm{Tr_{m+1}}(Z T_m).\end{aligned} \tag{27}$$

Our calculation of (26) is based on the approach of [6]. The following statement can be found in that paper.

Proposition 4.2. *Consider the rational function in u with values in the Hecke algebra $\mathcal{H}_m$ which is defined by*

$$Z_{m+1}(u) = \mathrm{Tr_{m+1}}\,(u - y_{m+1})^{-1}\,, \qquad y_{m+1} \in \mathcal{H}_{m+1},$$

where $\mathcal{H}_m$ is regarded as a subalgebra of $\mathcal{H}_{m+1}$. Then,

$$Z_{m+1}(u) = \frac{l\,Q + u - 1}{t\,u(u-1)} \left(\prod_{k=1}^{m} \frac{(u - y_k)^2}{(u - q^2 y_k)(u - q^{-2} y_k)} - \frac{(1 - l\,Q)(u-1)}{l\,Q + u - 1} \right), \tag{28}$$

where $l = q - q^{-1}$.

Proof. ¿From the definition of the Jucys-Murphy elements (2) we deduce the identity

$$\frac{1}{u - y_{m+1}} = T_m \frac{1}{u - y_m} T_m^{-1} + \frac{1}{u - y_m} \left(T_m^{-1} + \frac{lu}{(u - y_{m+1})} \right) \frac{l y_m}{(u - y_m)}. \tag{29}$$

Applying the map Tr_{m+1} to both sides of (29) and using (27) we get

$$\frac{(u - q^2 y_m)(u - q^{-2} y_m)}{(u - y_m)^2} Z_{m+1}(u) = Z_m(u) + \frac{l(1 - Ql) y_m}{(u - y_m)^2}.$$

For all $k = 1, \ldots, m+1$ introduce the function $\bar{Z}_k(u)$ by

$$Z_k(u) = \bar{Z}_k(u) + (Q - l^{-1}) u^{-1}.$$

This gives the relation

$$\bar{Z}_{m+1}(u) = \frac{(u - y_m)^2}{(u - q^2 y_m)(u - q^{-2} y_m)} \bar{Z}_m(u).$$

Solving this recurrence relation with the initial condition

$$\bar{Z}_1(u) = \mathrm{Tr}_1 (u - y_1)^{-1} - (Q - l^{-1}) u^{-1} = \frac{t\,Q + u - 1}{t\,u(u-1)},$$

we come to (28). ∎

The normalization factor $f(\lambda)$ defined in (19) and the quantum dimension (26) turn out to be related as shown in the following proposition. As before, we let λ be a partition of n, and $\mathcal{T}$ a standard λ-tableau.

Proposition 4.3. *We have the relation*

$$f(\lambda) = \mathrm{qdim}\,\mathcal{T}\prod_{k=1}^{n}\sigma_k\left(Q+\frac{\sigma_k-1}{q-q^{-1}}\right)^{-1}.$$

Proof. Using (14) and (15) we get

$$\mathrm{Tr_n}(E_{\mathcal{T}}) = \mathrm{Tr_n}E_{\mathcal{T}}(u)\big|_{u=\sigma_n} = E_{\mathcal{U}}\,(u-\sigma_n)\mathrm{Tr_n}(u-y_n)^{-1}\big|_{u=\sigma_n}.$$

Using equations (28) and taking into account (12) we obtain

$$\begin{aligned}\mathrm{Tr_n}(E_{\mathcal{T}}) = \frac{1}{\sigma_n}&\left(Q+\frac{\sigma_n-1}{l}\right)E_{\mathcal{U}}\\ &\times\frac{u-\sigma_n}{u-1}\left(\prod_{k=1}^{n-1}\frac{(u-\sigma_k)^2}{(u-q^2\sigma_k)(u-q^{-2}\sigma_k)}-(u-1)\frac{1-lQ}{lQ+u-1}\right)\Bigg|_{u=\sigma_n}=\\ &\qquad\qquad\qquad\qquad\qquad\qquad=\frac{1}{\sigma_n}\left(Q+\frac{\sigma_n-1}{l}\right)E_{\mathcal{U}}F_n(\sigma_n).\end{aligned}$$

Applying the maps $\mathrm{Tr_k}$ consequently, we finally obtain

$$\mathrm{qdim}\,\mathcal{T} = \mathcal{T}r^n(E_{\mathcal{T}}) = \mathrm{Tr_1}\mathrm{Tr_2}\ldots\mathrm{Tr_n}(E_{\mathcal{T}}) = \prod_{m=1}^{n}\frac{1}{\sigma_m}\left(Q+\frac{\sigma_m-1}{l}\right)F_m(\sigma_m).$$

The statement now follows from Lemma 3.2. ■

The following corollary is immediate from Proposition 4.3.

Corollary 4.4. *The Ocneanu–Markov trace $\mathcal{T}r^n(E_{\mathcal{T}})$ of the idempotent $E_{\mathcal{T}}$ depends only on the shape λ of $\mathcal{T}$ and does not depend on $\mathcal{T}$.*

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